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A DISPLACEMENT METRIC FOR FINITE SETS OF RIGID BODY DISPLACEMENTS

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ABSTRACT

There are various useful metrics for finding the distance between two points in Euclidean space. Metrics for finding the distance between two rigid body locations[1] in Euclidean space depend on both the coordinate frame and units used. A metric independent of these choices is desirable. This paper presents a metric for a finite set of rigid body displacements. The methodology uses the principal frame (PF) associated with the finite set of displacements and the polar decomposition to map the homogenous transform representation of elements of the special Euclidean group SE(N-1) onto the special orthogonal group SO(N). Once the elements are mapped to SO(N) a bi-invariant metric can then be used. The metric obtained is thus independent of the choice of fixed coordinate frame i.e. it is left invariant. This metric has potential applications in motion synthesis, motion generation and interpolation. Three examples are presented to illustrate the usefulness of this methodology.

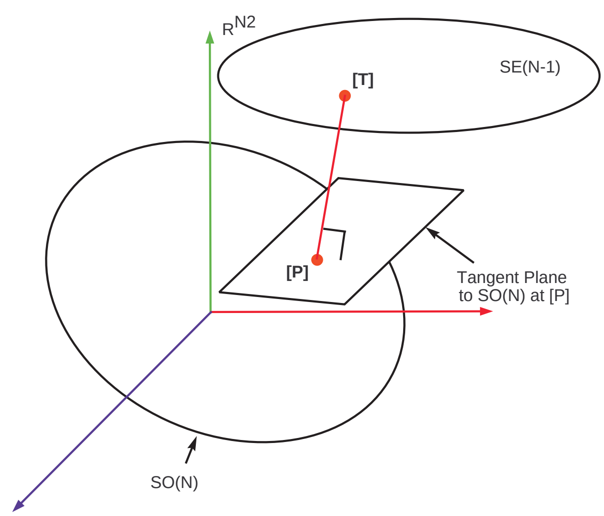

Figure 1. SE(N-1) to SO(N)

INTRODUCTION

A metric is used to measure the distance between two points in a set. There are various metrics for finding the distance between two points in Euclidean space. However, finding the distance between two locations of a rigid body is still the subject of ongoing research, see [1–9]. For two locations of a finite rigid body (either SE(2)-planar or SE(3)-spatial) all metrics yield a distance which is dependant upon the chosen fixed or moving frames of reference and the units used, see [2, 4]. But, a metric independent of these choices, referred to as bi-invariant, is desirable. Metrics independent of the choice of coordinate frames and the units used do exist on SO(N), see Larochelle [5]. One bi-invariant metric defined by Ravani and Roth [10] defines the distance between two orientations of a rigid body as the magnitude of the difference between

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[1]Location of a rigid body prescribes both its position and orientation.

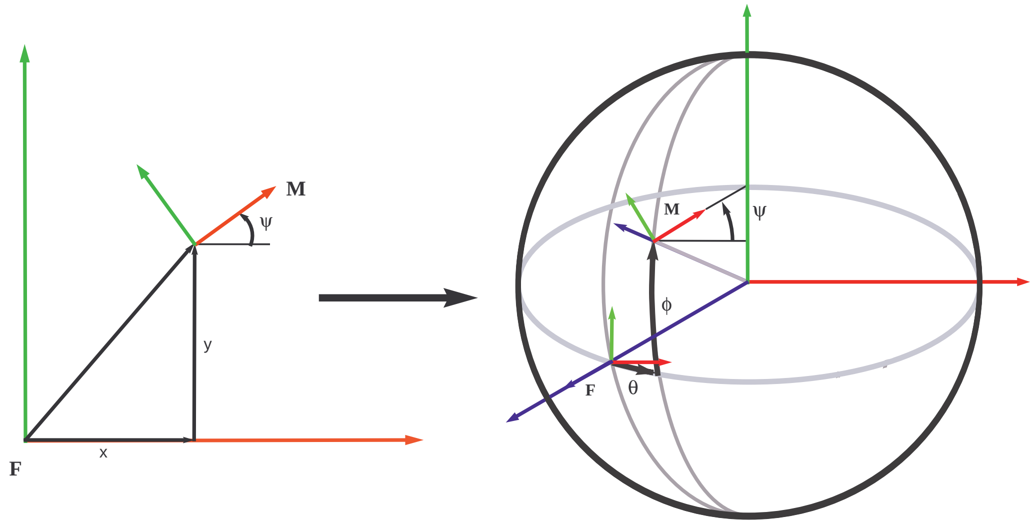

Figure 2. SE(2) to SO(3)

the associated quaternions. The techniques presented here are based on the polar decomposition (PD) of the homogenous transform representation of the elements of SE(N) and the principal frame (PF) associated with the finite set of rigid body displacements. The mapping of the elements of the special Euclidean group SE(N-1) to SO(N) yields hyperdimensional rotations that approximate the rigid body displacements. A conceptual representation of the mapping of SE(N-1) to SO(N) is shown in Figure 1. Once the elements are mapped to SO(N) distances can then be evaluated by using a bi-invariant metric on SO(N). In the planar case the elements of SE(2) are mapped onto the SO(3) as shown in Figure 2. The resulting PD based projection metric on SE(N-1) is left invariant (i.e. independent of the choice of fixed frame F).

METRIC ON SO(N)

The distance between elements in SO(N) can be determined by using the metric suggested by Larochelle [11]. The distance between two elements $[A_1]$ and $[A_2]$ in SO(N) can be defined by using the Frobenius norm as follows,

$$d = \|[I] - [A_2][A_1]^T\|_F \tag{1}$$

FINITE SETS OF LOCATIONS

Consider the case when a finite number of n displacements ($n \geq 2$) are given and we have to find the magnitude of these displacements. The displacements depend on the coordinate frame and the system of units chosen. In order to yield a left invariant metric we utilize a PF that is derived from a unit point mass model for a moving body as suggested by Larochelle [11]. This is done to yield a metric that is independent of the geometry of the moving body. The center of mass and the principal axes frame are unique for the system and invariant with respect to both the choice of fixed coordinate frames as well as the system of units [12, 13].

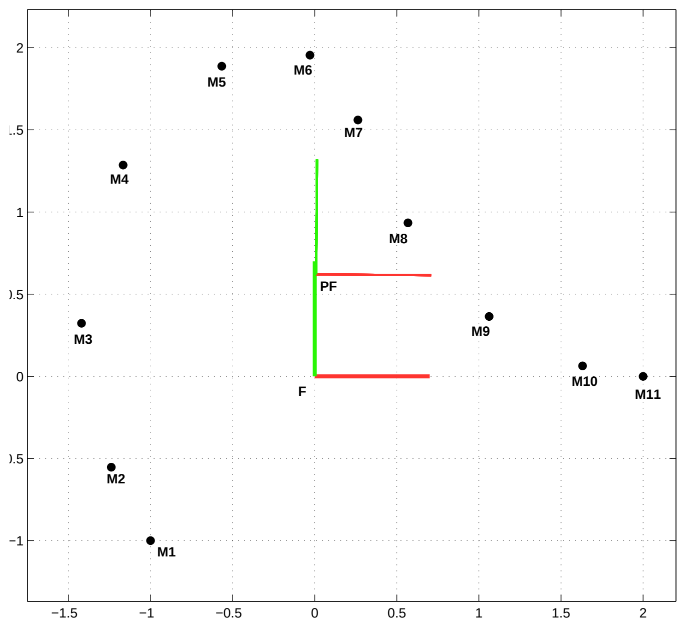

Figure 3. Unit Point Mass Model

The procedure for determining the center of mass $\vec{c}$ and the PF associated with the n prescribed locations is described below. A unit point mass is located at the origin of each location as shown in Figure 3.

$$\vec{c} = \frac{1}{n}\sum_{i=1}^{n}\vec{d_i} \tag{2}$$

where, $\vec{d_i}$ is the translation vector associated with the i^{th} location (i.e. the origin of the i^{th} location with respect to F).

The PF is defined such that its axes are aligned with the principal axes of the n point mass system and its origin is at the centroid $\vec{c}$. After finding the centroid of the system we determine the principal axes of the point mass system. The inertia tensor is,

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{3}$$

where the principal moments of inertia are defined by,

$$\begin{aligned} I_{xx} &= \sum_{i=1}^{n}(y_i^2 + z_i^2) \\ I_{yy} &= \sum_{i=1}^{n}(z_i^2 + x_i^2) \\ I_{zz} &= \sum_{i=1}^{n}(x_i^2 + y_i^2) \end{aligned} \tag{4}$$

the products of inertia are,

$$\begin{aligned} I_{xy} = I_{yx} &= -\sum_{i=1}^{n}(x_i y_i) \\ I_{xz} = I_{zx} &= -\sum_{i=1}^{n}(x_i z_i) \\ I_{yz} = I_{zy} &= -\sum_{i=1}^{n}(y_i z_i) \end{aligned} \tag{5}$$

and x_i, y_i, z_i are the components of $\vec{d_i}$. The principal frame is thus determined to be

$$[PF] = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where, $\vec{v_i}$ are the principal axes (eigenvectors) associated with the inertia tensor $[I]$, see Greenwood [12]. The directions of the vectors along the principal axes $\vec{v_i}$ are chosen such that the principal frame is a right handed system. However, Equation (6) does not uniquely define the PF since the eigenvectors $\vec{v_i}$ of the inertia tensor are not unique; both $\vec{v_i}$ and $-\vec{v_i}$ are eigenvectors associated with $[I]$. In order to resolve this ambiguity and yield a unique PF we choose to use the PF that is most closely aligned to F.

In the planar case the inertia tensor $[I]$ reduces to

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

and, the principal frame for the planar case reduces to a 3×3 matrix as shown:

$$[PF] = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{c} \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The eight different right handed PF's that are possible in the spatial case are given by,

$$\begin{array}{c} [\ \vec{v_1}\ \ \vec{v_2}\ \ \vec{v_3}\ ] \\ [\ \vec{v_2}\ \ -\vec{v_1}\ \ \vec{v_3}\ ] \\ [-\vec{v_1}\ \ -\vec{v_2}\ \ \vec{v_3}\ ] \\ [-\vec{v_2}\ \ \vec{v_1}\ \ \vec{v_3}\ ] \\ [\ \vec{v_2}\ \ \vec{v_1}\ \ -\vec{v_3}\ ] \\ [\ \vec{v_1}\ \ -\vec{v_2}\ \ -\vec{v_3}\ ] \\ [-\vec{v_2}\ \ -\vec{v_1}\ \ -\vec{v_3}\ ] \\ [-\vec{v_1}\ \ \vec{v_2}\ \ -\vec{v_3}\ ] \end{array}$$

In the planar case there are four possible orientations of the PF as seen in Figure 4.

$$\begin{array}{c} [\ \vec{v_1}\ \ \vec{v_2}\ ] \\ [\ \vec{v_2}\ \ -\vec{v_1}\ ] \\ [\ \vec{v_1}\ \ -\vec{v_2}\ ] \\ [-\vec{v_2}\ \ \vec{v_1}\ ] \end{array}$$

The PF that is most closely oriented to the fixed frame is chosen using the metric on SO(N) given in Equation (1).

MAPPING TO SO(N)

The unit disparity between translation and rotation is resolved by normalizing the translational terms in displacements. The displacements are normalized by choosing a characteristic length R. The characteristic length used, based upon the investigations reported in [5, 14], is $\frac{24L}{\pi}$, where L is the maximum translational component in the set of displacements at hand. Larger characteristic lengths result in an increase in the weight on the rotational terms whereas smaller ones result in an increase in weight on the translational terms. It was shown in [14] that

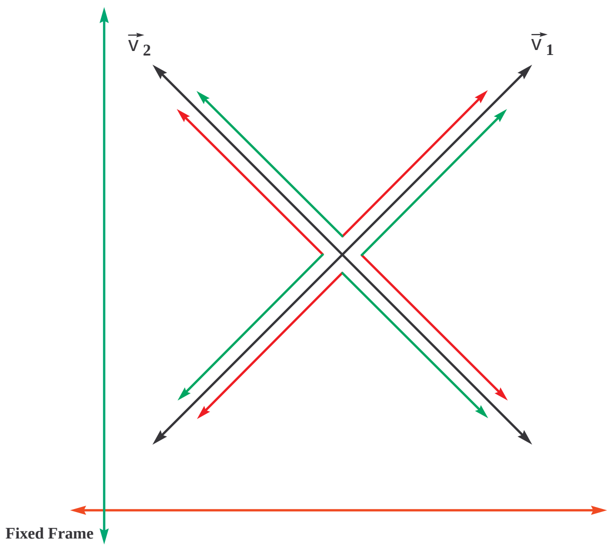

Figure 4. Four Possible Orientations for the PF

this characteristic length yields an effective balance between translational and rotational displacement terms for projection metrics.

The elements in SO(N) are derived from the polar decomposition of the homogenous transformations representing planar SE(2) or spatial SE(3) displacements. A number of iterative algorithms exist for the evaluation of the polar decomposition. Hingham described a method based upon Newtons method, see [15]. A simple and efficient iterative algorithm for the computation of the polar decomposition is shown by Dubrulle [16]. The algorithm produces mono-tonic convergence in the Frobenius norm that delivers an IEEE solution [17] in ~ 10 or fewer steps.

The elements SE(N) in the planar and spatial cases are represented by,

$$T_i = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{[R]} & \vec{t} \\ \hline 0 & 0 & 1 \end{array}\right] \tag{9}$$

and,

$$T_i = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{[R]} & \vec{t} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{10}$$

where $[R]$ represents the rotational component and $\vec{t}$ represents the translational component of the homogenous transformation of the locations. The scaled transformation matrices for the planar and spatial cases are thus obtained to be,

$$T_{i(scaled)} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{[R]} & \vec{t}/R \\ \hline 0 & 0 & 1 \end{array}\right] \tag{11}$$

and

$$T_{i(scaled)} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{[R]} & \vec{t}/R \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{12}$$

where, R represents the characteristic length used to resolve the unit disparity between rotation and translation. The scaled transformation matrices may then be mapped to SO(N) by using the Dubrulle algorithm for PD.

SUMMARY OF THE TECHNIQUE

For a set of n finite rigid body locations the steps to be followed are:

1. Determine the PF associated with the n locations.
2. Determine the relative displacements from PF to each of the n locations.
3. Determine the characteristic length R associated with the n displacements with respect to the PF and scale the translation terms in each by $1/R$.
4. Compute the projections of PF and each of the scaled relative displacements using the polar decomposition algorithm.
5. The magnitude of the displacement is defined as the distance from PF to the scaled relative displacement as computed via Equation (1). The distance between any two of the n locations is similarly computed by the application of Equation (1) to the projected scaled relative displacements.

EXAMPLE: ELEVEN PLANAR LOCATIONS

Consider the rigid body guidance problem proposed by J. Michael McCarthy, U.C. Irvine for the 2002 ASME International Design Engineering Technical Conferences held in Montreal, Quebec and listed in [18]. The 11 planar locations are listed in Table 1 and the origins of the coordinate frames with the respect to the fixed reference frame F are shown in Figure 3. The centroid of the system is determined to be $\vec{c} = [0.0094\ 0.6199]^T$. Next, the principal axes directions are determined. The principal

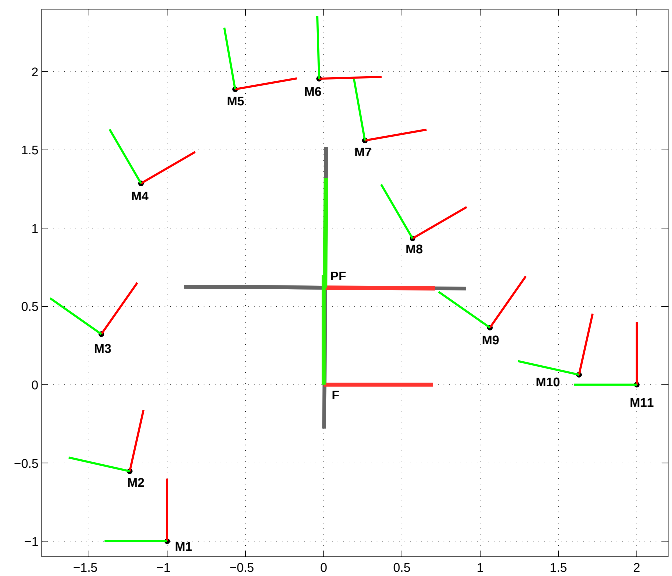

Figure 5. Principal Frame for Eleven Desired Locations

axes directions and $\overrightarrow{c}$ are used to determine the principal frame.

$$[PF] = \begin{bmatrix} 1.0000 & 0.0067 & 0.0094 \\ -0.0067 & 1.0000 & 0.6199 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix} \tag{13}$$

The eleven locations are now determined with respect to the PF and the maximum translational component is found to be 1.9947 and the resulting characteristic length $R = \frac{24L}{\pi}$ = 15.239. The 11 locations are then scaled by the characteristic length in order to find the distance to the principal frame. The magnitude of each of the displacements with respect to the PF is listed in Table 1. The distance between any two of the locations is computed by the application of Equation (1) to the projected scaled relative displacements. For example the distance between location #1 and location #2 was found to be 0.3115.

EXAMPLE: FOUR SPATIAL LOCATIONS

Consider the rigid body guidance problem investigated by Larochelle [11]. The 4 spatial locations are listed in Table 2 with respect to the fixed reference frame F and are shown in Figure 6. The principal frame is determined to be

$$[PF] = \begin{bmatrix} 0.8061 & 0.5692 & -0.1617 & 0.7500 \\ -0.5916 & 0.7807 & -0.2012 & 1.5000 \\ 0.0117 & 0.2578 & 0.9661 & 0.4375 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix} \tag{14}$$

Table 1. Eleven Planar Locations

#	x	y	α (deg)	Mag.
1	−1.0000	−1.0000	90.0000	2.0076
2	−1.2390	−0.5529	77.3621	1.7762
3	−1.4204	0.3232	55.0347	1.3165
4	−1.1668	1.2858	30.1974	0.7483
5	−0.5657	1.8871	10.0210	0.2644
6	−0.0292	1.9547	1.7120	0.0807
7	0.2632	1.5598	10.0300	0.2606
8	0.5679	0.9339	30.1974	0.7464
9	1.0621	0.3645	55.0346	1.3159
10	1.6311	0.0632	77.3620	1.7762
11	2.0000	0.0000	90.0000	2.0078

Table 2. Four Desired Locations

#	x	y	z	θ	ϕ	ψ	Mag.
1	0.00	0.00	0.00	0.0	0.0	0.0	0.95
2	0.00	1.00	0.25	15.0	15.0	0.0	1.24
3	1.00	2.00	0.50	45.0	60.0	0.0	2.21
4	2.00	3.00	1.00	45.0	80.0	0.0	2.44

The maximum translational component is found to be 2.0276 and the associated characteristic length is R = 15.4899. The magnitude of each of the displacements with respect to the PF is listed in Table 2.

EXAMPLE: TEN SPATIAL LOCATIONS

Consider the rigid body guidance problem investigated by Larochelle [11]. The 10 spatial locations with respect to the the fixed reference frame F are listed in Table 3 and shown in Figure 7. The principal frame is given by,

$$[PF] = \begin{bmatrix} 0.756 & 0.655 & 0.000 & 5.500 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ 0.655 & -0.756 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix} \tag{15}$$

The maximum translational component L is found to be 6.7256 and the associated characteristic length is $R = \frac{24L}{\pi}$ = 51.3795.

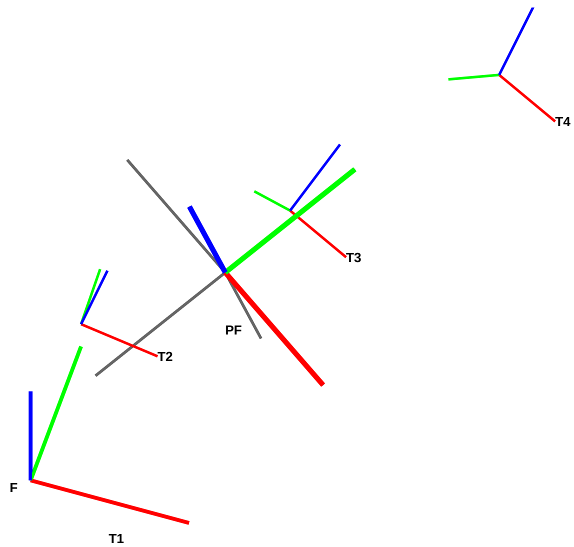

Figure 6. Principal Frame for Four Desired Locations

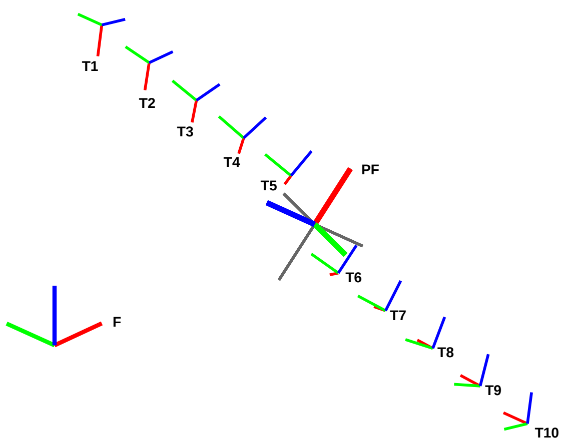

Figure 7. Principal Frame for Ten Desired Locations

The distance from the first location to the principal frame was found to be 2.7488. The distance between location #1 and location #2 was found to be 0.3485.

CONCLUSIONS

We have developed a metric for a finite set of rigid body displacements which uses a mapping of the special Euclidean group SE(N-1). This technique is based on embedding SE(N-1) into SO(N) via the polar decomposition of the homogeneous transform representation of SE(N-1). To yield a useful metric for a finite set of displacements appropriate for design applications, the principal frame and the characteristic length are used. A bi-invariant metric on SO(N) is then used to measure the distance between any two displacements in SE(N-1). A detailed algorithm for the application of this method was presented and illustrated by three examples. This technique has potential applications in mechanism synthesis and robot motion planning.

Table 3. Ten Desired Locations.

#	x	y	z	*Long* (θ)	*Lat* (ϕ)	*Roll* (ψ)
1	1.00	0.00	5.00	100	0.00	0.00
2	2.00	0.00	4.00	90	0.00	10.00
3	3.00	0.00	3.00	80	0.00	20.00
4	4.00	0.00	2.00	70	0.00	30.00
5	5.00	0.00	1.00	60	0.00	40.00
6	6.00	0.00	-1.00	50	0.00	50.00
7	7.00	0.00	-2.00	40	0.00	60.00
8	8.00	0.00	-3.00	30	0.00	70.00
9	9.00	0.00	-4.00	20	0.00	80.00
10	10.00	0.00	-5.00	10	0.00	90.00

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